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Coset construction for duals of non-relativistic CFTs

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Abstract: We systematically analyze backgrounds that are holographic duals to non-relativistic CFTs, by constructing them as cosets of the Schrödinger group and variants thereof. These cosets G/H are generically non-reductive and we discuss in generality how a metric on such spaces can be determined from a non-degenerate H-invariant symmetric two-form. Applying this to the $d = 2$ Schrödinger algebra, we reproduce the five-dimensional backgrounds proposed as duals of fermions at unitarity, and under reasonable physical assumptions, we demonstrate uniqueness of this background. The proposed gravity dual of the Lifshitz fixed-point, for which Galileian symmetry is absent, also fits into this organizational scheme and uniqueness of this background can also be shown.

Keywords: Gauge-gravity correspondence, AdS-CFT Correspondence, Space-Time Symmetries

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1 Introduction and summary

Variants of the AdS/CFT correspondence, which provide gravity duals of non-relativistic gauge theories, could potentially be of great importance in order to describe strongly coupled, scale-invariant condensed matter systems. Examples of such systems are fermions at unitarity and theories at Lifshitz-like fixed points.

The simplest proposals for gravity duals to non-relativistic theories were put forward in [1–3]. One natural question that arises is, whether these spaces are homogeneous and whether they are a comprehensive list of such backgrounds. In this note, we want to address this question, and find an organizational principle to study duals to non-relativistic theories as homogeneous spaces. The particular cosets involved in this construction are rather non-standard, in that they do not generically yield symmetric spaces. We provide a framework for studying such backgrounds and then demonstrate that under certain physical assumptions, the metrics found in [1–3] are unique. It will be interesting to extend this to the supercosets for the super-Schrödinger algebras and study the corresponding backgrounds for superstring theory.

We begin in section 2 by discussing the general theory of invariant metrics on cosets, in particular focusing on their existence for general cosets that are not necessarily reductive. This will be important, as the background found in [1, 2] are non-reductive cosets of the Schrödinger algebra [4, 5]. In order to construct the metric on these cosets, the key ingredient is the existence of a nondegenerate symmetric two-form, that is invariant under the denominator group, as in [6]. We apply this general theory to spaces with Schrödinger symmetry in section 3 and to the dual of the Lifshitz fixed point in section 4, and demonstrate how these are unique under certain physical assumptions on the subgroup.

2 General considerations on cosets

2.1 Homogeneous spaces and invariant two-forms

Consider a coset (homogeneous space) $M = G/H$, where G is a Lie group and H is a Lie subgroup of G. Let us denote the corresponding Lie algebras by $\mathfrak{g}$ and $\mathfrak{h}$, respectively. For each $g \in \mathfrak{g}$, let us denote the corresponding element of $\mathfrak{g}/\mathfrak{h}$ by $[g]$. As a vector space, we can always decompose

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}\,, \tag{2.1}$$

but there is an ambiguity in the choice of $\mathfrak{m}$. One can impose various compatibility conditions of the Lie algebra structure with this linear space decomposition. The coset M is called a reductive coset if there is a choice of $\mathfrak{m}$ such that it is $\mathrm{ad}(\mathfrak{h})$-invariant, i.e.

$$[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}\,. \tag{2.2}$$

If we impose in addition that

$$[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}\,, \tag{2.3}$$

then M is a symmetric space, which is equivalent to the existence of a $\mathbb{Z}_2$ grading, such that $\deg(\mathfrak{h}) = 0$, $\deg(\mathfrak{m}) = 1$, which is compatible with the Lie algebra structure.

We wish to construct a G-invariant metric on the homogenous space M. If $\mathfrak{g}$ is semi-simple, then the Killing form is non-degenerate and induces a G-invariant metric on M. In the case of degenerate Killing form, the existence of such a G-invariant metric is not guaranteed, however the following proposition gives a useful criterion:

Proposition ([7], Proposition X.3.1)
There is a one-to-one correspondence between G-invariant indefinite Riemannian metrics $\mathcal{G}$ on $M = G/H$ and $\mathrm{Ad}(H)$-invariant non-degenerate symmetric bilinear forms Ω on $\mathfrak{g}/\mathfrak{h}$. When H is connected, $\mathrm{Ad}(H)$-invariance of Ω reduces to $\mathrm{ad}(\mathfrak{h})$-invariance, meaning that

$$\Omega([h, [t_1]], [t_2]) + \Omega([t_1], [h, [t_2]]) = 0\,, \tag{2.4}$$

for any $h \in \mathfrak{h}$, or equivalently, written in terms the structure constants of the Lie algebra,

$$\Omega_{[m][n]} f_{[k]p}{}^{[m]} + \Omega_{[k][m]} f_{[n]p}{}^{[m]} = 0\,, \tag{2.5}$$

where $[m], [n] \cdots$ denotes generators of $\mathfrak{m}$ and p is an $\mathfrak{h}$-index. The structure constants $f_{[k]p}{}^{[m]}$ are well-defined since $\mathfrak{h}$ is a subalgebra and different representatives of the coset element $[k]$ give the same answer.

For our purposes, it is important to know the explicit relation between Ω and the metric. Let us first define a metric on the identity element $[e] = eH$ of $M = G/H$. We want to define a metric $\mathcal{G}(X_1, X_2)$, where X_1 and X_2 are elements of T_eM. Recall that $T_eG = \mathfrak{g}$ and likewise, $T_eM = \mathfrak{g}/\mathfrak{h}$. Therefore, X_1 and X_2 can be identified with elements $[t_1]$ and $[t_2]$ of $\mathfrak{g}/\mathfrak{h}$, respectively. Under this identification, the explicit correspondence between Ω and $\mathcal{G}$ is given by

$$\mathcal{G}(X_1, X_2)_{[e]} = \mathcal{G}([t_1], [t_2])_{[e]} = \Omega([t_1], [t_2])\,, \qquad t_1\,,\ t_2 \in \mathfrak{g}\,. \tag{2.6}$$

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Next we need to define a metric at an arbitrary point $[g]$ of $M = G/H$. By the G-action on the whole manifold, it is possible to translate the metric at the origin to any other point. Choose an arbitrary representative g of $[g]$. Then left-multiplication by g^{-1} yields the map $L_{g^{-1}} : M \to M$. From this map we have an induced map

$$[J_g] := (L_{g^{-1}})_* : \ T_{[g]}M \to T_{[e]}M = \mathfrak{g}/\mathfrak{h}\,, \tag{2.7}$$

which is a $\mathfrak{g}/\mathfrak{h}$-valued one-form on $\mathfrak{m}$. This is the Maurer-Cartan (MC) one-form. Similarly, we can define $J_g : T_gG \to T_eG = \mathfrak{g}$.

For two vector fields $Y_1, Y_2 \in T_{[g]}M$, define the metric $\mathcal{G}(Y_1, Y_2)_{[g]}$ at point $[g]$ to be

$$\mathcal{G}(Y_1, Y_2)_{[g]} = \mathcal{G}([J_g](Y_1), [J_g](Y_2))_{[e]}\,. \tag{2.8}$$

When $\mathfrak{g}$ is embedded into $\mathfrak{gl}_N$ and g takes matrix values (which we will assume throughout this paper), the MC one-form can be written as

$$[J_g] = [g^{-1}dg]\,. \tag{2.9}$$

In the discussion above we chose a particular representative g for $[g]$. If we choose another representative gh with $h \in H$, we have

$$[J_{gh}] = [(gh)^{-1}d(gh)] = [\mathrm{ad}(h)(g^{-1}dg)], \tag{2.10}$$

where we have used the relation $[h^{-1}dh] = 0$ as an element of $\mathfrak{g}/\mathfrak{h}$. This relation, together with the definition of the metric in (2.8) and the H-invariance condition of (2.4), tells us that the metric (2.8) is independent of the choice of the representative of $[g]$. This shows the well-definedness of the metric.

Several comments are now in order:

1. The structure constants $f_{[k]p}{}^{[m]}$ we used above are in general different from the structure constants $f_{kp}{}^{m}$ of $\mathfrak{g}$. They are equivalent only for reductive cosets (2.2).

2. Whenever $\mathfrak{g}$ is semi-simple, the Killing form is non-degenerate and provides a natural candidate for Ω. In many instances that will be of interest to us, the Killing form is degenerate and the invariant non-degenerate two-form we use is different from the Killing form.

3. Reductiveness is a natural notion for Riemannian cosets: if G is an isometry group of a Riemannian metric on G/H and if H is connected, then G/H is automatically reductive [7]. However, in Lorentzian signature this is in general not true and some of the examples we discuss below are indeed non-reductive. We therefore do not impose either condition (2.2) and (2.3) in the following discussions.

4. We emphasize again that in general neither existence nor uniqueness of such a two-form Ω is guaranteed. For some coset G/H, Ω does not exist, and for others there exists a family of such invariant two-forms, as we shall see exemplified below.

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5. A homogeneous space is mathematically defined as a space M with a transitive action of G, meaning for any two points x, x' of M we can find an element $g_{x,x'}$ of G such that $g_{x,x'}.x = x'$. From this condition it follows that M is written as a coset G/G_x, where G_x is the stabilizer at point x. If we choose a different point x', $G_{x'}$ and G_x are in general different, but belong to the same conjugacy class. This means that classification of cosets of the form G/H for a given G reduces to the two problems: first to classify conjugacy classes of its subgroups and second to classify the non-degenerate invariant two-forms of the subgroup.

In summary, a homogeneous space is characterized by the data (G, H, Ω), where Ω is a $\mathfrak{h}$-invariant nondegenerate symmetric two-form specifying the G-invariant metric on the coset space G/H. We will apply this general discussion to the cases of interest in the context of non-relativistic conformal theories.

2.2 Explicit coordinate description of cosets

In the previous section, we used a coordinate invariant formalism. However, in order to derive explicit forms of the metrics it is often useful to go to a particular coordinate frame. For that purpose, we first fix a linear space decomposition (2.1), as well as a basis $t_m, t_n, \ldots$ for $\mathfrak{h}$ and $t_p, t_q, \ldots$ for $\mathfrak{m}$. Then we parametrize an element $[g] \in G/H$ by

$$[g] = [\exp(x_m t_m) \exp(x_n t_n) \ldots ..] \quad (\text{modulo } H). \tag{2.11}$$

Of course, the expression in (2.11) is far from unique. For example, another possible parametrization is

$$[g] = \left[\exp\left(\sum_{m,n,..} x_m t_m \right) \right]. \tag{2.12}$$

These are just different choices of coordinates on G/H and are related by coordinate transformations. We will thus choose a convenient expression in each of the subsequent discussions.

The MC one-form $J_g = g^{-1} dg$ can now be computed explicitly and decomposed according to (2.1):

$$J_g = e_m t_m + e_p t_p. \tag{2.13}$$

In this notation, the metric defined in (2.6), (2.8) is written as

$$G = \Omega^{mn} e_m e_n \,, \tag{2.14}$$

namely, e_m are nothing but vielbeine, which get contracted with Ω. If we choose a different representation gh for $[g]$,

$$J_{gh} = \mathrm{Ad}(h)(J_g), \tag{2.15}$$

and the vielbeine mix among themselves, which shows that H is a symmetry of the vielbeine.

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3 The Schrödinger algebra and cosets

3.1 The Schrödinger algebra

The Schrödinger algebra $\mathfrak{Sch}_d$ in $d+1$ dimensions [4, 5] has generators J^{ij} (spatial rotations), P^i (spatial translations), H (Hamiltonian), G^i (Galileian boosts), D (dilatation) and C (special conformal transformations). One can consider a central extension $\widetilde{\mathfrak{Sch}}_d$ of this algebra by the mass operator M. The non-vanishing commutation relations are

$$\begin{aligned}
&[J^{ij}, J^{kl}] = -\delta^{ik}J^{jl} + \delta^{il}J^{jk} - \delta^{jl}J^{ik} + \delta^{jk}J^{il}\,,\\
&[J^{ij}, P^k] = -\delta^{ik}P^j + \delta^{jk}P^i\,, \qquad [J^{ij}, G^k] = -\delta^{ik}G^j + \delta^{jk}G^i\,,\\
&[H, G^i] = P^i\,, \qquad [D, G^i] = -G^i\,, \qquad [C, P^i] = -G^i\,, \qquad [D, P^i] = P^i\,,\\
&[D, H] = 2H\,, \qquad [H, C] = -D\,, \qquad [D, C] = -2C\,,
\end{aligned} \tag{3.1}$$

as well as the central extension

$$\left[P^i, G^j\right] = \delta^{ij}M\,. \tag{3.2}$$

This algebra is a subalgebra of conformal algebra, as was observed in [1, 8, 9].

3.2 Subalgebras and two-forms for $d = 2$

Let us consider the case $d = 3$. In this case, denoting $J^{12} = J$, the algebra is[1]

$$\begin{aligned}
&[J, P^1] = -P^2\,, \quad [J, P^2] = P^1\,, \quad [J, G^1] = -G^2\,, \quad [J, G^2] = G^1\,,\\
&[H, G^i] = P^i\,, \quad [D, G^i] = -G^i\,, \quad [C, P^i] = -G^i\,, \quad [D, P^i] = P^i\,,\\
&[D, H] = 2H\,, \quad [H, C] = -D\,, \quad [D, C] = -2C\,, \quad [P^i, G^j] = \delta^{ij}M\,.
\end{aligned} \tag{3.3}$$

For $d = 2$ there is unfortunately no satisfactory classification result for subalgebras.[2] However, in addition to being a subalgebra, there are various physically motivated conditions, that are naturally imposed upon $\mathfrak{h}$:

Assumption 1 (No translation condition). $\mathfrak{h}$ does not contain P^i.

This is natural because P^i will be realized as infinitesimal translations in the geometry, and should not be included in the stabilizer of a point on the homogeneous space G/H. Another condition we impose is:

Assumption 2 (Lorentz subgroup condition). $\mathfrak{h}$ contains J^{ij} and G^i.

This condition is needed because we want to respect d-dimensional local Lorentz symmetry, which is crucial for the equivalence principle of general relativity.[3]

Although our methods apply to Schrödinger cosets in arbitrary dimensions, let us concentrate on the case of $\dim G/H = 5$ and $\dim H = 4$. This is the case discussed recently

[1] We use the same symbole H to denote the Hamiltonian of $\mathfrak{Sch}$ and the denominator subgroup of the coset. We hope no confusion will arise.

[2] Conjugacy classes of subalgebras of the Schrödinger algebra are classified in [10].

[3] In the literature, stronger constraints are imposed on G [11], although for our purposes the Lorentz subgroup condition is strong enough.

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in the literature, which in the context of the non-relativistic AdS/CFT correspondence is conjectured to be dual to $(2+1)$-dimensional non-relativistic conformal field theories [1, 2].

If we impose the above two assumptions then $\mathfrak{h}$ is spanned by J, G^1, G^2 and one more generator, and the possible choices are

$$\begin{aligned}
\mathfrak{h}_{(1)} &= \langle J, G^1, G^2, \alpha C + \beta M + \gamma D \rangle \qquad (\alpha \neq 0)\,, \\
\mathfrak{h}_{(2)} &= \langle J, G^1, G^2, \beta M + \gamma D \rangle \quad (\beta \neq 0)\,, \qquad (3.4) \\
\mathfrak{h}_{(3)} &= \langle J, G^2, G^2, D \rangle. \qquad (3.5)
\end{aligned}$$

The $\mathrm{Ad}(H)$-invariant two-forms are obtained by solving for Ω in (2.5). This requires in particular a specification of the basis of generators of the complement $\mathfrak{m}_i$ of each subalgebra. Define

$$\begin{aligned}
\mathfrak{m}_{(1)} &= \langle H, P^1, P^2, M, D \rangle\,, \\
\mathfrak{m}_{(2)} &= \langle H, P^1, P^2, C, D \rangle\,, \\
\mathfrak{m}_{(3)} &= \langle H, P^1, P^2, C, M \rangle\,, \qquad (3.6)
\end{aligned}$$

Let us consider in detail the case $\mathfrak{h}_{(1)}$. Assuming that $\beta \neq 0$, the structure constants relevant for (2.5) are

$$f_{[i}J^{[j]} = \begin{pmatrix} 0&0&0&0&0\\ 0&0&+1&0&0\\ 0&-1&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}, \qquad f_{[i}{}_{\alpha C+\beta M+\gamma D}{}^{[j]} = \begin{pmatrix} -2\gamma&0&0&0&-\alpha\\ 0&-\gamma&0&0&0\\ 0&0&-\gamma&0&0\\ 0&0&0&0&0\\ 0&0&0&2\beta&2\gamma \end{pmatrix},$$

$$f_{[i}G^1{}^{[j]} = \begin{pmatrix} 0&+1&0&0&0\\ 0&0&0&1&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}, \qquad f_{[i}G^2{}^{[j]} = \begin{pmatrix} 0&0&+1&0&0\\ 0&0&0&0&0\\ 0&0&0&1&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}. \qquad (3.7)$$

Then solving for Ω in (2.5) yields for $\gamma \neq 0$ the resulting two-form is degenerate. For $\alpha, \beta \neq 0$ and $\gamma = 0$ there exists a non-degenerate two-form

$$\Omega^{(1)}_{[i][j]} = \begin{pmatrix} \Omega_{HH}&0&0&-\Omega_{PP}&0\\ 0&\Omega_{PP}&0&0&0\\ 0&0&\Omega_{PP}&0&0\\ -\Omega_{PP}&0&0&0&0\\ 0&0&0&0&-2\frac{\beta}{\alpha}\Omega_{PP} \end{pmatrix}, \qquad \alpha, \beta \neq 0\,,\ \gamma = 0\,. \qquad (3.8)$$

Parameterizing the coset elements as

$$g = e^{x_H H} e^{x_M M} e^{x_i P^i} e^{x_D D}\,, \qquad (3.9)$$

then the vielbeine are given by

$$e_H = e^{2x_D} dx_H, \qquad e_M = dx_M, \qquad e_i = e^{x_D} dx^i\,, \qquad e_D = dx_D\,. \qquad (3.10)$$

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One may take $\Omega_{PP} = 1$ by using the overall scaling. Then $\sigma := \Omega_{HH}$ can be freely chosen, and by suitable coordinate transformation, we can set $2\beta/\alpha = 1$, which yields

$$ds^2 = r^2(-2dx^+dx^- + dx^i dx^i) + \frac{dr^2}{r^2} - \sigma r^4 (dx^+)^2, \tag{3.11}$$

where we defined

$$e^{x_D} = r, \qquad x_H = x^+, \qquad x_M = x^- . \tag{3.12}$$

For $\sigma = 0$, this is the DLCQ of AdS_5 [12, 13]. Although the metric looks locally the same as that of AdS_5, the presence of M, which commutes with all other generators, means that the eigenvalue of M is quantized and the corresponding direction, namely the x^+-direction, is compactified. The deformation term proportional to σ is nothing but the deformation term that was also observed to be present in [1, 2]. One way of understanding the appearance of this deformation term is the null Melvin twist [14–16], but from the viewpoint of the coset this is simply a deformation parameter of the invariant two-form.[4]

This coset, constructed from $\mathfrak{h}_{(1)} = \{J, G^i, \alpha C + \beta M\}\,(\alpha, \beta \neq 0)$ is an interesting example of a non-reductive coset. The examples of non-reductive cosets are scarce in the literature, and in four dimensions and less. A classification of Lorentzian non-reductive homogeneous spaces in four and less dimensions appears in [18].

In a similar fashion, one can analyze the case of $\mathfrak{h}_{(2)}$ and $\mathfrak{h}_{(3)}$, and in both instances we have verified the non-existence of a non-degenerate invariant two-forms. This implies that under the above assumptions, the metric (3.11) is unique:

Uniqueness. Under the Assumptions 1 and 2 above, the 5d coset of $\widetilde{\mathfrak{Sch}_2}$ is unique, and the metric is given by (3.11).

One way to escape this uniqueness theorem is to abandon assumption 2. Although in such cases the rotation symmetry or Galilean boost symmetry is broken as a symmetry of the local frames, they still exist as symmetries of the background, and these can potentially become useful in the future study of non-relativistic AdS/CFT correspondence.

Relaxing assumption 2, there are two further choices for subalgebras:

$$\begin{aligned} \mathfrak{h}_{(4)} &= \langle G^1, G^2, C, D + \alpha M \rangle\,, \\ \mathfrak{h}_{(5)} &= \langle H, C, D \rangle \oplus \langle \alpha J + \beta M \rangle\,. \end{aligned} \tag{3.13}$$

Their complements can be chosen as

$$\begin{aligned} \mathfrak{m}_{(4)} &= \langle H, J, P^1, P^2, M \rangle\,, \\ \mathfrak{m}_{(5)} &= \langle P^1, P^2, G^1, G^2, J \rangle\,, \end{aligned} \tag{3.14}$$

and then straightforward computation shows that for $\mathfrak{h}_{(4)}$ there do not exist any non-degenerate two-forms. For $\mathfrak{h}_{(5)}$ with $\beta \neq 0$, we obtain

$$\Omega = 2\Omega_{P_1,G_2}(e_{P_1}e_{G_2} - e_{P_2}e_{G_1}) + \Omega_{JJ}e_J^2\,. \tag{3.15}$$

[4] Applying this method to the pp-wave case [17], which is reductive, would also yield a deformation term. However, this can be removed by a coordinate transformation and has no physical significance.

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Parameterizing the coset elements as

$$g = e^{x_i P_i} e^{y_i G_i} e^{x_J J}, \tag{3.16}$$

the invariant one-forms are

$$\begin{aligned} e_{P_1} &= dx_1 \cosh x_J - dx_2 \sinh x_J, & e_{P_2} &= dx_1 \sinh x_J + dx_2 \cosh x_J, \\ e_{G_1} &= dy_1 \cosh x_J - dy_2 \sinh x_J, & e_{G_2} &= dy_1 \sinh x_J + dy_2 \cosh x_J, \\ e_J &= -\gamma(dx_1 y_1 + dx_2 y_2) + dx_J\,, \end{aligned} \tag{3.17}$$

where $\gamma = \frac{\alpha}{\beta}$. Equations (3.15) and (3.17) yields a metric with Schrödinger symmetry:

$$ds^2 = (dx_1 dy_2 - dx_2 dy_1)\cosh 2x_J + \sigma(dx_J - \gamma(y_1 dx_1 + y_2 dx_2))^2 \tag{3.18}$$

with $\sigma \neq 0$. Unfortunately, the signature of this spacetime is $(2,3)$, and as such does not seem to be suitable for applications in AdS/CFT. Therefore, even by relaxing the conditions in assumption 2, the background (3.11) seems to be unique.

3.3 The cases of $z \neq 2$

In a non-relativistic spacetime, we can scale time and space differently:

$$t \to \lambda^z t, \qquad x \to \lambda x\,, \tag{3.19}$$

where the parameter z is called the dynamical exponent. The discussion so far corresponds to the case $z = 2$, and we are now going to consider the case with arbitrary dynamical exponent $z \neq 2$. The algebra (which we call $\widetilde{\mathfrak{Sch}}_{d,z}$) is given by

$$\begin{aligned} &[J^{ij}, J^{kl}] = -\delta^{ik} J^{jl} + \delta^{il} J^{jk} - \delta^{jl} J^{ik} + \delta^{jk} J^{il}\,, \\ &[J^{ij}, P^k] = -\delta^{ik} P^j + \delta^{jk} P^i\,, && [J^{ij}, G^k] = -\delta^{ik} G^j + \delta^{jk} G^i\,, \\ &[H, G^i] = P^i\,, \qquad [P^i, G^j] = \delta^{ij} M\,, && [D^z, H] = zH\,, \\ &[D^z, P^i] = P^i\,, \qquad [D^z, G^i] = (1-z) G^i\,, && [D^z, M] = (2-z) M\,. \end{aligned} \tag{3.20}$$

Note that C is broken in the case with $z \neq 2$.

Under the two assumptions of the previous section, a natural coset candidate is

$$\mathfrak{h} = \{G^i, J^{ij}\}\,, \qquad \mathfrak{m} = \{H, M, P^i, D^z\}\,. \tag{3.21}$$

The invariant two-form associated to this choice is

$$\Omega = \begin{pmatrix} \Omega_{HH} & -\Omega_{PP} & 0 & 0 & \Omega_{HD} \\ -\Omega_{PP} & 0 & 0 & 0 & 0 \\ 0 & 0 & \Omega_{PP} & 0 & 0 \\ 0 & 0 & 0 & \Omega_{PP} & 0 \\ \Omega_{DH} & 0 & 0 & 0 & \Omega_{DD} \end{pmatrix}. \tag{3.22}$$

A group element of G/H is represented by $g = \mathrm{e}^{x_H H} \mathrm{e}^{x_M M} \mathrm{e}^{x_i P^i} \mathrm{e}^{x_D D^z}$, and the vielbeine are

$$e_H = \mathrm{e}^{z x_D} dx_H\,, \qquad e_M = \mathrm{e}^{(2-z) x_D} dx_M\,, \qquad e_i = \mathrm{e}^{x_D} dx_i\,, \qquad e_D = dx_D\,. \tag{3.23}$$

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The metric is, up to coordinate transformations, given by

$$\begin{aligned} ds^2 &= -2e_H e_M + e_i^2 + e_D^2 + \sigma e_H^2 \\ &= r^2(-2dx^+dx^- + dx^i dx^i) + \frac{dr^2}{r^2} + \sigma r^{2z}(dx^+)^2\,, \end{aligned} \tag{3.24}$$

where we have identified as $x_H = x^+, x_M = x^-, \mathrm{e}^{x_D} = r$. When $\sigma = 0$, this again yields the DLCQ of AdS, and σ is a deformation term [2] similar to the one discussed in the previous section.

3.4 A comment on super-cosets

It would be interesting to consider the super-cosets related to non-relativistic AdS/CFT backgrounds. The most interesting example of super coset is represented by a subalgebra of psu(2,2|4) [8, 9]. This symmetry is known to be realized by the background consisting of the metric of DLCQ of AdS_5 times S^5. That is, the x^--compactification breaks the relativistic conformal symmetry to the Schrödinger and the 16 superconformal symmetries are broken to 8.

For 24 supercharges there should be no deformation term [15].[5] However, it seems difficult to see this from the coset description, since the argument for the metric is unaltered. This result is not so surprising since the on-shell condition of supergravity is not taken into account, as well as the fact that the number of supersymmetries is not maximal. In particular, the presence of the B-field, which breaks the symmetry of the metric, is not included in our argument.

4 Gravity dual of Lifshitz fixed points and cosets

In the previous section we fix $\mathfrak{g}$ to be $\widetilde{\mathfrak{Sch}}$ and considered various choices of subgroups $\mathfrak{h}$. Consider now cosets, where $\mathfrak{g}$ is a subalgebra of the Schrödinger algebra.

First we need to address the question of which subgroup of $\widetilde{\mathfrak{Sch}}$ we should take as $\mathfrak{g}$. Let us first discuss the case $z = 2$. One possibility is to search for an interesting subgroup which does not contain M. This is because in the discussion above M corresponds to a x^+-directions, and this is the origin of the difficulties associated with DLCQ in the dual CFT.

Since $[P^i, G^j] = \delta^{ij} M$, we have to remove either P^i or G^i. Since we want to keep translation invariance, let us remove G^i. Then again since $[C, P^i] = G^i$, we also need to remove C. The the remaining generators H, D, P^i, J span a subgroup of $\widetilde{\mathfrak{Sch}}$. In the case $z \neq 2$ (3.20), we can consider the same algebra. We thus consider $\mathfrak{g} = \langle J^{ij}, P^i, H, D^z, M\rangle$. Consider the case $d = 2$, since higher dimensional case are similar. If we are going to consider a 4d coset of $\mathfrak{g}$, $\mathfrak{h}$ is one-dimensional and assumption 2 above uniquely determines

[5] In [19] a class of supersymmetric Schrödinger backgrounds is discussed, however, these are not homogeneous spaces.

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$\mathfrak{h}$ to be $\mathfrak{h} = \langle J^{ij} \rangle$.[6] Then the relevant commutation relations are

$$[J, P^1] = -P^2\,, \qquad [J, P^2] = P^1\,, \qquad [D^z, H] = zH\,, \qquad [D^z, P^i] = P^i\,. \tag{4.1}$$

Let $\mathfrak{m} = \langle H, P^1, P^2, D^z \rangle$. Then solving for (2.5) we obtain

$$\Omega_{[i][j]} = \begin{pmatrix} \Omega_{HH} & 0 & 0 & \Omega_{HD} \\ 0 & \Omega_{P^1P^1} & 0 & 0 \\ 0 & 0 & \Omega_{P^1P^1} & 0 \\ \Omega_{HD} & 0 & 0 & \Omega_{DD} \end{pmatrix}. \tag{4.2}$$

In this case, all the deformation parameters in the invariant two-forms are removed by coordinate transformations, and the resulting metric is

$$ds^2 = -r^{2z}dt^2 + r^2 dx_i^2 + \frac{dr^2}{r^2}. \tag{4.3}$$

This is precisely the background in [3], which is the candidate gravity dual of the Lifshitz fixed point. Again, similar arguments as in the previous section seem to show that this is the unique 4d coset of this group even when the assumption 2 is relaxed.

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References

[1] D.T. Son, *Toward an AdS/cold atoms correspondence: a geometric realization of the Schroedinger symmetry*, *Phys. Rev.* **D 78** (2008) 046003 [arXiv:0804.3972] [SPIRES].

[2] K. Balasubramanian and J. McGreevy, *Gravity duals for non-relativistic CFTs*, *Phys. Rev. Lett.* **101** (2008) 061601 [arXiv:0804.4053] [SPIRES].

[3] S. Kachru, X. Liu and M. Mulligan, *Gravity duals of Lifshitz-like fixed points*, *Phys. Rev.* **D 78** (2008) 106005 [arXiv:0808.1725] [SPIRES].

[4] C.R. Hagen, *Scale and conformal transformations in galilean-covariant field theory*, *Phys. Rev.* **D 5** (1972) 377 [SPIRES].

[6] In this case, Galilean symmetry is not inlucded in $\mathfrak{h}$ since it is broken from the outset. However, the theory discussed here has a "doubled" Galilean symmetry acting on particles and anti-particles in an opposite way.

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[5] U. Niederer, *The maximal kinematical invariance group of the free Schrödinger equation*, *Helv. Phys. Acta* **45** (1972) 802 [SPIRES].

[6] C.R. Nappi and E. Witten, *A WZW model based on a nonsemisimple group*, *Phys. Rev. Lett.* **71** (1993) 3751 [hep-th/9310112] [SPIRES].

[7] S. Kobayashi and K. Nomizu, *Foundations of differential geometry. Vol. I*, in *Wiley Classics Library*, John Wiley & Sons Inc., New York U.S.A. (1996), reprint of the 1963 original, A Wiley-Interscience Publication.

[8] M. Sakaguchi and K. Yoshida, *Super Schrödinger in super conformal*, *J. Math. Phys.* **49** (2008) 102302 [arXiv:0805.2661] [SPIRES].

[9] M. Sakaguchi and K. Yoshida, *More super Schrödinger algebras from $PSU(2,2|4)$*, *JHEP* **08** (2008) 049 [arXiv:0806.3612] [SPIRES].

[10] G. Burdet, J. Patera, M. Perrin and P. Winternitz, *Sous-algèbres de Lie de l'algèbre de Schrödinger*, *Ann. Sci. Math. Québec* **2** (1978), 81.

[11] S. Weinberg, *Quasiriemannian theories of gravitation in more than four-dimensions*, *Phys. Lett.* **B 138** (1984) 47 [SPIRES].

[12] W.D. Goldberger, *AdS/CFT duality for non-relativistic field theory*, *JHEP* **03** (2009) 069 [arXiv:0806.2867] [SPIRES].

[13] J.L.F. Barbon and C.A. Fuertes, *On the spectrum of nonrelativistic AdS/CFT*, *JHEP* **09** (2008) 030 [arXiv:0806.3244] [SPIRES].

[14] C.P. Herzog, M. Rangamani and S.F. Ross, *Heating up galilean holography*, *JHEP* **11** (2008) 080 [arXiv:0807.1099] [SPIRES].

[15] J. Maldacena, D. Martelli and Y. Tachikawa, *Comments on string theory backgrounds with non- relativistic conformal symmetry*, *JHEP* **10** (2008) 072 [arXiv:0807.1100] [SPIRES].

[16] A. Adams, K. Balasubramanian and J. McGreevy, *Hot spacetimes for cold atoms*, *JHEP* **11** (2008) 059 [arXiv:0807.1111] [SPIRES].

[17] R.R. Metsaev, *Type IIB Green-Schwarz superstring in plane wave Ramond-Ramond background*, *Nucl. Phys.* **B 625** (2002) 70 [hep-th/0112044] [SPIRES].

[18] M.E. Fels and A.G. Renner, *Non-reductive homogeneous pseudo-Riemannian manifolds of dimension four*, *Canad. J. Math.* **58** (2006), 282

[19] S.A. Hartnoll and K. Yoshida, *Families of IIB duals for nonrelativistic CFTs*, *JHEP* **12** (2008) 071 [arXiv:0810.0298] [SPIRES].

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